

New fractional entangling transform and its quantum mechanical correspondence

Shuang Xu (徐双), Liyun Hu (胡利云)*, and Jiehui Huang (黄接辉)

Center for Quantum Science and Technology, Jiangxi Normal University, Nanchang 330022, China

*Corresponding author: hlyun@jxnu.edu.cn

Received November 20, 2014; accepted December 22, 2014; posted online February 24, 2015

In this Letter, a new fractional entangling transformation (FrET) is proposed, which is generated in the entangled state representation by a unitary operator $\exp\{i\theta(a\hat{b}^\dagger + a^\dagger b)\}$ where $a(b)$ is the Bosonic annihilate operator. The operator is actually an entangled one in quantum optics and differs evidently from the separable operator, $\exp\{i\theta(a^\dagger a + b^\dagger b)\}$, of complex fractional Fourier transformation. The additivity property is proved by employing the entangled state representation and quantum mechanical version of the FrET. As an application, the FrET of a two-mode number state is derived directly by using the quantum version of the FrET, which is related to Hermite polynomials.

OCIS codes: 270.0270, 070.2575.

doi: 10.3788/COL201513.030801.

As a generalization of the Fourier transform, the fractional Fourier transform (FrFT) has attracted much attention and is used widely to design optical systems and optimize holographic storage efficiency^[1-5], whose applications range from filter design and signal analysis to phase retrieval and pattern recognition. Is there correspondence between the classical-optical transform and quantum-optical transform (unitary operator)? By using the Dirac's symbolic method and the technique of integration within an ordered product (IWOP), the corresponding relation between them has been bridged^[6]. For example, the wavelet transform is transformed into a matrix element of a squeezing-translating operator between the state vector corresponding to the given mother wavelet and the state to be transformed by using this method^[7]. This indicates that we may not only find a new quantum mechanical unitary operator corresponding to the classical transform, but we also propose some new classical-optical transforms from the point of quantum mechanics. In this Letter, on the basis of complex FrFT, we shall introduce a new FrET in a 2D case, whose quantum mechanical operator corresponds to an entangling operator.

A brief review of the FrFT and complex FrFT is presented next. The FrFT in one dimension of θ order is defined as

$$\mathcal{F}_\theta[f(x)] = \int_{-\infty}^{\infty} K_\theta(x, y)f(y)dy, \quad (1)$$

where the integral kernel function $K_\theta(x, y)$ is

$$K_\theta(x, y) = \sqrt{\frac{e^{i(\frac{\pi}{2}-\theta)}}{2\pi \sin \theta}} \exp\left\{-i\frac{x^2 + y^2}{2 \tan \theta} + \frac{ixy}{\sin \theta}\right\}. \quad (2)$$

If we define $f(x) \equiv \langle x|f\rangle$ and $K_\theta(x, y) = \langle y|S_\theta|x\rangle$ where $|y\rangle$ and $|x\rangle$ are coordinate Eigenvectors, $X|x\rangle = x|x\rangle$, then multiplying the function $K_\theta(x, y)$ by the ket $\int_{-\infty}^{\infty} dy|y\rangle$ and bra $\int_{-\infty}^{\infty} dx\langle x|$ from left and right, and using the IWOP technique^[8] to perform the integration, we can finally get

$$\begin{aligned} S_\theta &= \int_{-\infty}^{\infty} dx dy |y\rangle K_\theta(x, y) \langle x| =: \exp\{(e^{i\theta} - 1)a^\dagger a\}: \\ &= \exp\{i\theta a^\dagger a\}, \end{aligned} \quad (3)$$

where in the last step, we have used the formula $\exp\{fa^\dagger a\} =: \exp\{(e^f - 1)a^\dagger a\}:^{\text{[9,10]}}$. Thus the FrFT can be described, in quantum-mechanical language, as

$$\mathcal{F}_\theta[f(x)] = \int_{-\infty}^{\infty} \langle y|S_\theta|x\rangle \langle x|f\rangle dx = \langle y|S_\theta|f\rangle, \quad (4)$$

where the completeness relation of coordinate representation is used

$$\int_{-\infty}^{\infty} dx |x\rangle \langle x| = 1. \quad (5)$$

From Eqs. (3) and (4), one can clearly see that the unitary operator S_θ corresponding to the 1D FrFT is just the rotation operator, and the FrFT is actually the matrix element of S_θ in coordinate state $\langle y|$ and transformed vector $|f\rangle$ under the quantum mechanical version of the FrFT. Employing this version, it is easy and direct to discuss some properties of the FrFT.

As an extension, the complex FrFT is introduced by defining the following transform^[11]

$$\begin{aligned} \mathcal{F}_\theta[f](\eta') &= \int_{-\infty}^{\infty} \frac{d^2\eta}{\pi} \mathcal{K}_C(\eta', \eta) f(\eta), \\ \mathcal{K}_C(\eta', \eta) &= \frac{e^{i(\frac{\theta-\pi}{2})}}{2 \sin \theta} \exp\left[\frac{i(|\eta'|^2 + |\eta|^2)}{2 \tan \theta} - \frac{i(\eta'^*\eta + \eta^*\eta')}{2 \sin \theta}\right]. \end{aligned} \quad (6)$$

In a similar way, using the IWOP technique and the completeness relation of the entangled state representation^[9]

$$\int_{-\infty}^{\infty} \frac{d^2\eta}{\pi} |\eta\rangle \langle \eta| = 1. \quad (7)$$

where $|\eta\rangle = \eta_1 + i\eta_2\rangle$ is the Eigenvector of commutable operators: relative position $X_1 - X_2$ and the total momentum $P_1 + P_2$, i.e., $(X_1 - X_2)|\eta\rangle = \sqrt{2}\eta_1|\eta\rangle$, $(P_1 + P_2)|\eta\rangle = \sqrt{2}\eta_2|\eta\rangle$, and

$$|\eta\rangle = \exp\left[-\frac{1}{2}|\eta|^2 + \eta a^\dagger - \eta^* b^\dagger + a^\dagger b^\dagger\right]|00\rangle. \quad (8)$$

We can present Eq. (6) as the quantum version of

$$\mathcal{F}_\theta[f](\eta') = \langle\eta'| \exp[-i\theta(a^\dagger a + b^\dagger b)]|f\rangle. \quad (9)$$

Actually, from Eqs. (3) and (9) we see that the corresponding unitary operators of the 1D and complex FrFT are the (product of) rotation operators. Then an interesting question naturally arises. Is there a unitary entangling operator corresponding to a kind of complex FrFT? The answer is affirmative. Next we shall propose such a fractional transform.

Regarding the fractional entangling transform, if we replace the integration kernel [Eq. (6)] with

$$\begin{aligned} \mathcal{K}_C(\eta', \eta) &= \frac{1}{2 \sin \theta} \\ &\times \exp\left[\frac{i(\eta'^2 + \eta^2 + \eta'^{*2} + \eta^{*2})}{4 \tan \theta} - \frac{i(\eta\eta' + \eta^*\eta'^*)}{2 \sin \theta}\right], \end{aligned} \quad (10)$$

then to what kind of transform does it belong? Is it still a fractional FrFT? Actually, Eq. (10) just corresponds to the integration kernel of the fractional entangling transform. Letting $\mathcal{K}_C = \langle\eta'| U_\theta|\eta\rangle$, and using the completeness relation of entangled state representation [Eq. (7)], we can express the operator U_θ as

$$U_\theta = \int_{-\infty}^{\infty} \frac{d^2\eta d^2\eta'}{\pi^2} |\eta'\rangle \mathcal{K}_C(\eta', \eta) \langle\eta|. \quad (11)$$

Furthermore, use of Eq. (8) and the normally ordering form of vacuum project operator $|00\rangle\langle 00| = \exp\{-a^\dagger a - b^\dagger b\}$:^[9,12] as well as the IWOP technique, we can derive

$$\begin{aligned} U_\theta &= \frac{1}{2 \sin \theta} \int \frac{d^2\eta d^2\eta'}{\pi^2} : \exp\left\{-\frac{1}{2}(|\eta'|^2 + |\eta|^2) \right. \\ &\quad \left. + \eta' a^\dagger - \eta'^* b^\dagger + \eta^* a - \eta b + \frac{i(\eta^2 + \eta'^2 + \eta'^{*2} + \eta^{*2})}{4 \tan \theta} \right. \\ &\quad \left. - \frac{i(\eta\eta' + \eta^*\eta'^*)}{2 \sin \theta} + ab + a^\dagger b^\dagger - a^\dagger a - b^\dagger b\right\} \\ &=: \exp[(\cos \theta - 1)(a^\dagger a + b^\dagger b) + i(ab^\dagger + a^\dagger b) \sin \theta]: \end{aligned} \quad (12)$$

where we have used the integration formula (Appendix A of Ref. [13])

$$\begin{aligned} &\int \frac{d^2z}{\pi} \exp(-\zeta|z|^2 + \xi z + \eta z^* + f z^2 + g z^{*2}) \\ &= \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp\left[\frac{\zeta\xi\eta + \xi^2g + \eta^2f}{\zeta^2 - 4fg}\right]. \end{aligned} \quad (13)$$

Equation (12) is the normally ordering form of operator U_θ . In order to see clearly the quantum mechanical correspondence of Eq. (12), we first derive the transform relation $U_\theta a U_\theta^\dagger$. Using the normally ordering form of U_θ and the completeness relation of the coherent state $\int d^2\alpha |\alpha\rangle\langle\alpha|/\pi = 1$, we can directly derive

$$\begin{aligned} U_\theta a U_\theta^\dagger &= \frac{\partial}{\partial \tau} \exp[(a \cos \theta - ib \sin \theta)\tau] \Big|_{\tau=0} \\ &= a \cos \theta - ib \sin \theta, \\ U_\theta b U_\theta^\dagger &= b \cos \theta - ia \sin \theta, \end{aligned} \quad (14)$$

where we have used the IWOP technique and Eq. (13). Thus, making the partial derivative for parameter θ and using the transform in Eq. (14) we can obtain

$$\begin{aligned} \frac{\partial}{\partial \theta} U_\theta &= i(b^\dagger U_\theta a + a^\dagger U_\theta b) \cos \theta - (a^\dagger U_\theta a + b^\dagger U_\theta b) \sin \theta \\ &= [i(b^\dagger U_\theta a U_\theta^\dagger + a^\dagger U_\theta b U_\theta^\dagger) \cos \theta \\ &\quad - (a^\dagger U_\theta a U_\theta^\dagger + b^\dagger U_\theta b U_\theta^\dagger) \sin \theta] U_\theta \\ &= i(ab^\dagger + a^\dagger b) U_\theta, \end{aligned} \quad (15)$$

which indicates that

$$U_\theta = \exp\{i\theta(ab^\dagger + a^\dagger b)\}. \quad (16)$$

Obviously, U_θ is a unitary operator ($U_\theta^\dagger = U_\theta^{-1}$). It is interesting to note that U_θ is actually a two-mode squeezing operator (or entangling operator). Thus we name the unitary operator U_θ the fractional entangling operator which is evidently different from the one of complex FrFT in Eq. (9). In fact, the unitary operator U_θ can be realized experimentally by the interactions correspond to positive detuning when the resonator frequency is larger than the laser frequency in the optomechanical case (a nonlinear process)^[14].

Regarding the additivity of fractional transform, an important feature of the FrFT is the additivity property, i.e., $\mathcal{F}_\theta \mathcal{F}_{\theta'} = \mathcal{F}_{\theta+\theta'}$. In order to prove this property of the fractional entangling transform, using Eq. (16), we can put the transform corresponding to the kernel [Eq. (10)] as

$$\begin{aligned} \mathcal{F}_\theta[f](\eta') &= \int_{-\infty}^{\infty} \frac{d^2\eta}{\pi} \mathcal{K}_C(\eta', \eta) f(\eta) \\ &= \int_{-\infty}^{\infty} \frac{d^2\eta}{\pi} \langle\eta'| U_\theta|\eta\rangle \langle\eta|f\rangle \\ &= \langle\eta'| \exp\{i\theta(ab^\dagger + a^\dagger b)\}|f\rangle, \end{aligned} \quad (17)$$

which is just the quantum version of the FrET. This indicates that the FrET just corresponds to the matrix

element of unitary operator $\exp\{i\theta(ab^\dagger + a^\dagger b)\}$ in the entangled state representation $\langle\eta'|\$ and the signal vector $|f\rangle$.

Using the quantum version of FrFT and the completeness relation of the entangled state [Eq. (7)], it is easy to see that

$$\begin{aligned}\mathcal{F}_{\theta+\alpha}[f(\eta)] &= \langle\eta'|\exp\{i(\theta+\alpha)(ab^\dagger + a^\dagger b)\}|f\rangle \\ &= \int_{-\infty}^{\infty} \frac{d^2\eta''}{\pi} \langle\eta'|\exp\{i\theta(ab^\dagger + a^\dagger b)\}|\eta''\rangle \\ &\quad \times \int_{-\infty}^{\infty} \frac{d^2\eta}{\pi} \langle\eta''|\exp\{i\alpha(ab^\dagger + a^\dagger b)\}|\eta\rangle f(\eta) \\ &= \int_{-\infty}^{\infty} \frac{d^2\eta''}{\pi} \langle\eta'|\exp\{i\theta(ab^\dagger + a^\dagger b)\}|\eta''\rangle \mathcal{F}_\alpha[f(\eta)] \\ &= \mathcal{F}_\theta \mathcal{F}_\alpha[f(\eta)].\end{aligned}\quad (18)$$

Thus the FrET actually satisfies the fractional additivity. The proof is clear and concise by using the representation $\langle\eta'|\$ and the quantum version of the FrET.

Regarding applications, the quantum version [Eq. (17)] of the FrET can help us to derive the FrET of some wave functions conveniently. In this Letter, we consider the two-mode number state as the signal vector, $|f\rangle = |m, n\rangle = a^{\dagger m} b^{\dagger n} / \sqrt{m!n!} |00\rangle$. Noting the coherent state representation of the number state $|m\rangle = 1/\sqrt{m!} (\partial^m / \partial \alpha^m) |\alpha\rangle|_{\alpha=0}$, where $|\alpha\rangle = \exp\{\alpha a^\dagger\} |0\rangle$ is the unnormalized coherent ($\langle 0|\alpha\rangle = 1$), thus

$$\begin{aligned}f(\eta) &= \frac{e^{-\frac{1}{2}|\eta|^2}}{\sqrt{m!n!}} \frac{\partial^{m+n}}{\partial \alpha^m \partial \beta^n} \exp[\eta^* \alpha - \eta \beta + \alpha \beta] \Big|_{\alpha, \beta=0} \\ &= \frac{i^{m+n} e^{-\frac{1}{2}|\eta|^2}}{\sqrt{m!n!}} H_{m,n}(-i\eta^*, i\eta),\end{aligned}\quad (19)$$

where $H_{m,n}(x, y)$ are two-variable Hermite polynomials whose mother function is defined by^[15]

$$H_{m,n}(x, y) = \frac{\partial^{m+n}}{\partial t^m \partial \tau^n} \exp[-t\tau + tx + \tau y] \Big|_{t=\tau=0}.\quad (20)$$

On the other hand, note the transform relation $U_\theta a^\dagger U_\theta^\dagger = a^\dagger \cos \theta + ib^\dagger \sin \theta$, and $U_\theta b^\dagger U_\theta^\dagger = b^\dagger \cos \theta + ia^\dagger \sin \theta$, it is easy to get

$$\begin{aligned}U_\theta |m, n\rangle &= \frac{1}{\sqrt{m!n!}} (a^\dagger \cos \theta + ib^\dagger \sin \theta)^m (b^\dagger \cos \theta + ia^\dagger \sin \theta)^n |00\rangle \\ &= \frac{1}{\sqrt{m!n!}} \frac{\partial^{m+n}}{\partial \alpha^m \partial \beta^n} e^{a^\dagger(\alpha \cos \theta + i\beta \sin \theta) + b^\dagger(i\alpha \sin \theta + \beta \cos \theta)} |00\rangle \Big|_{\alpha, \beta=0} \\ &= \frac{1}{\sqrt{m!n!}} \frac{\partial^{m+n}}{\partial \alpha^m \partial \beta^n} |\bar{\alpha}, \bar{\beta}\rangle \Big|_{\alpha, \beta=0},\end{aligned}\quad (21)$$

where $|\bar{\alpha}, \bar{\beta}\rangle$ ($\bar{\alpha} = \alpha \cos \theta + i\beta \sin \theta$, $\bar{\beta} = i\alpha \sin \theta + \beta \cos \theta$) are also unnormalized coherent states. Thus, using Eqs. (17) and (19) the FrET of the Hermite Gaussian function [Eq. (19)] can be calculated as

$$\begin{aligned}\mathcal{F}_\theta[f(\eta)] &= \langle\eta'|\ U_\theta |m, n\rangle \\ &= \frac{e^{-\frac{1}{2}|\eta'|^2}}{\sqrt{m!n!}} \frac{\partial^{m+n}}{\partial \alpha^m \partial \beta^n} \exp[\eta'^* \bar{\alpha} - \eta' \bar{\beta} + \bar{\alpha} \bar{\beta}] \Big|_{\alpha, \beta=0} \\ &= \sum_{l=0}^{\min(m,n)} \left(i \sqrt{\frac{i \sin 2\theta}{2}} \right)^{m+n-2l} \frac{\sqrt{n!m!} e^{-\frac{1}{2}|\eta'|^2} \cos^l 2\theta}{4^l l!(n-l)!(m-l)!} \\ &\quad \times H_{m-l} \left(\frac{\bar{\eta}}{i\sqrt{2i \sin 2\theta}} \right) H_{n-l} \left(\frac{\bar{\eta}^*}{i\sqrt{2i \sin 2\theta}} \right),\end{aligned}\quad (22)$$

where $\bar{\eta} = \eta'^* \cos \theta - i\eta' \sin \theta$ and we have used the mother function formula of single-variable Hermite polynomials

$$\begin{aligned}H_n(x) &= \frac{\partial^n}{\partial t^n} \exp(2xt - t^2) \Big|_{t=0}, \\ \frac{d}{dx^l} H_n(x) &= \frac{2^l n!}{(n-l)!} H_{n-l}(x).\end{aligned}\quad (23)$$

Equation (22) is just the FrET of two-mode number state which is related to the single-variable Hermite polynomials.

In conclusion, based on the complex FrFT, we successively propose a new kind of fractional transform (the FrET) by replacing the integration kernel with a new one. Employing the IWOP technique and the entangled state representation, the unitary operator corresponding to the FrET is just a entangling operator and is evidently different from that in the complex FrFT. The additivity property, an important fractional feature, is easily proved by using the matrix element expression and the completeness relation of entangled state representation in the framework of quantum mechanics. We should point out that we not only find the quantum mechanical correspondence (unitary operator) of the classical optical transform, but also develop the classical one from the viewpoint of quantum optics. That is to say, it is possible that some other transforms can be presented by using different quantum mechanical unitary operators or representations. In our derivation, the IWOP technique plays an important role.

This work was supported by the National Natural Science Foundation of China (Grant Nos. 11264018 and 11174118), the Natural Science Foundation of Jiangxi Province of China (Grant No. 20132BAB212006), and the Research Foundation of the Education Department of Jiangxi Province of China (No. GJJ14274), and the Degree and Postgraduate Education Teaching Reform Project of Jiangxi Province (No. JXYJG-2013-027).

References

1. N. C. Pard and J. W. Fleischer, Opt. Lett. **36**, 2551 (2011).
2. J. Chen, Y. Zhu, B. Liu, W. Wei, N. Wang, and J. Zhang, Chin. Opt. Lett. **11**, 053003 (2013).
3. T. Wu and Y. Liu, Chin. Opt. Lett. **11**, 021702 (2013).

4. Y. Chen, Y. Sui, and H. Yang, *Chin. Opt. Lett.* **11**, 082201 (2013).
5. A. Stern and Y. Rivenson, *Chin. Opt. Lett.* **12**, 060022 (2014).
6. H. Y. Fan and L. Y. Hu, *Front. Phys.* **7**, 261 (2012).
7. H. Y. Fan and H. L. Lu, *Opt. Lett.* **31**, 407 (2006).
8. H. Y. Fan, H. L. Lu, and Y. Fan, *Ann. Phys.* **321**, 480 (2006).
9. H. Y. Fan, *Representation and Transformation Theory in Quantum Mechanics* (Scientific and Technical, 1997), in Chinese.
10. L. Hu and Z. Zhang, *Chin. Opt. Lett.* **10**, 082701 (2012).
11. H. Y. Fan, *Opt. Lett.* **28**, 2177 (2003).
12. W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, 1973).
13. R. R. Puri, *Mathematical Methods of Quantum Optics* (Springer, 2001).
14. S. Sivakumar, *Phys. Rev. A* **83**, 035802 (2011).
15. A. Wüenshe, *J. Comp. Appl. Math.* **133**, 665 (2001).